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Survival probability of a single resonance

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Abstract

An exact single-level resonance formula for the survival probability $S(t)$ in the full time interval, that depends only on the resonance energy ϵ_r and the decay width Γ_r and fulfils time-reversal invariance, is used to discuss the non-exponential contributions to decay. At short times the formula behaves as $S(t) \approx 1 - ct^{1/2}$ with c a constant, whereas at long times it behaves as $S(t) \approx dt^{-3}$, d being a constant. With the time expressed in lifetime units, the onset of non-exponential decay is given at short times by $\tau_S \approx 4/[\pi(R^2 + R + 1/4)]$ and at long times by $\tau_L \approx 5.41 \ln(R) + 12.25$, where $R = \epsilon_r/\Gamma_r$. The predictions of the formula are compared with numerical examples and some experimental results searching for non-exponential contributions to decay.

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1. Introduction

The exponential decay law has been very successful in the description of the time evolution of decay in quantum systems as follows from work originated in the early days of quantum mechanics [1, 2]. The pioneering work of Khalfin [3], however, and subsequent theoretical work have pointed to an approximate validity of the exponential decay law. Since the energy spectrum is bounded from below, deviations from purely exponential decay are expected at very short and very long times compared with the lifetime of the system. The experimental verification of non-exponential decay has remained elusive for many decades. After years of intense effort [4, 5] the first detected deviation from the exponential decay law was reported recently in the short time regime in a tunnelling experiment involving an artificial system [6]. On the other hand, in the long time regime no observation of a deviation of the exponential decay law has been reported so far.

The description of the non-exponential contribution to decay seems to depart from the simplicity of the purely exponential decay law [1], where two quantities seem to suffice to describe it: ϵ_r, that represents the energy of the decaying fragment, and the width Γ_r, that represents the decay rate and provides an estimate of the lifetime t_l of the system through the well known relationship $t_l = \hbar/\Gamma_r$. It is the purpose of this paper to derive an expression for the survival probability of a single resonance that is valid in the full time interval and hence includes the non-exponential contributions to decay. As in the case of purely exponential decay we also assume that the initial state decays solely through the single resonance. We model the single-level resonance by a pole structure, consisting of a pair of complex poles on the momentum k-plane, that guarantees that the survival probability fulfils time-reversal invariance.

This paper is organized as follows. In section 2 we derive the survival amplitude for a single resonance as well as expressions for its short- and long time behaviour. Section 3 deals with an analysis of the survival probability. In particular we obtain expressions for the onset to non-exponential decay in the short- and long time regimes and provide numerical examples of these regimes. In section 4 we make a comparison with a numerical result for the delta-shell potential, characterized by many resonance terms [7, 8], obtaining an excellent agreement. We also apply the single-level formula to nuclear systems employed in the search of non-exponential contributions [4, 5], showing the impossibility of observing non-exponential decay in the time intervals considered in such experiments. Finally section 5 presents some concluding remarks.

2. Single-level resonance decay formula

The survival amplitude is defined as

$$A(t) = \int_0^\infty \psi^*(r,0)\psi(r,t)\,\mathrm{d}r. \tag{1}$$

The above expression gives the probability amplitude that at time t the decaying system remains in the initial state $\psi(r,0)$. The survival probability is therefore given by

$$S(t) = |A(t)|^2. \tag{2}$$

Here for the sake of simplicity we refer to zero angular momentum. It is convenient to write equation (1) in terms of the retarded time-dependent Green function $g(r,r';t)$ as

$$A(t) = \int_0^\infty \mathrm{d}r \int_0^\infty \mathrm{d}r'\,\psi(r,0)^* g(r,r';t)\psi(r',0). \tag{3}$$

Furthermore one may write $g(r,r';t)$ in terms of the outgoing Green function $G^+(r,r';k)$, through the Laplace transform,

$$g(r,r';t) = \frac{\mathrm{i}}{2\pi}\int_c G^+(r,r';k)\mathrm{e}^{-\mathrm{i}k^2t}2k\,\mathrm{d}k \tag{4}$$

where c represents an integration contour along the first quadrant of the complex k-plane and our units are $\hbar = 2m = 1$. Using equation (4) in (3) one may write the survival amplitude in the form

$$A(t) = \frac{\mathrm{i}}{2\pi}\int_c A(k)\mathrm{e}^{-\mathrm{i}k^2t}\,\mathrm{d}k \tag{5}$$

where $A(k)$ is given by

$$A(k) = 2k\int_0^\infty \mathrm{d}r\int_0^\infty \mathrm{d}r'\,\psi(r,0)G^+(r,r';k)\psi(r',0). \tag{6}$$

The connection between decay and the complex poles of the propagator goes back to Peierls [9]. It is well known from considerations on causality that the poles of the propagator are seated on the lower half of the complex k-plane and that time-reversal invariance requires that complex poles come in pairs. For each pole at $k_r = \alpha_r - \mathrm{i}\beta_r$ with $\alpha_r, \beta_r > 0$ there is another pole k_{-r}, situated symmetrically with respect to the imaginary axis, that is, $k_{-r} = -k_r^*$. Hence a correct description of a resonance must involve both k_r and $-k_r^*$. Similarly if ρ_r represents the residue at a complex pole of the propagator then $\rho_{-r} = \rho_r^*$.

In our model we assume that the single-level resonance is described entirely by a pair of complex poles k_r and $-k_r^*$. Assuming that $A(k) \to 0$ as $|k| \to \infty$ along all directions on the complex k-plane allows us to write $A(k)$ for a single resonance as

$$A(k) = \frac{D_r}{k - k_r} + \frac{D_r^*}{k + k_r^*} \tag{7}$$

where D_r is the residue at the pole k_r and involves a factor $2k_r$ multiplied by a double integration over space of the residue of the propagator and the initial state. Since there are no singularities on the upper half of the k-plane one may modify the contour c in equation (5) into a straight line going from $-\infty$ to ∞ and then by substitution in that expression of equation (7) one may write[3]

$$A(t) = D_r M(k_r, t) + D_r^* M(-k_r^*, t) \tag{8}$$

where the M-functions above follow from the definition [8],

$$M(q, t) \equiv \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}k^2 t}}{k - k_n} \, \mathrm{d}k = \frac{1}{2} \mathrm{e}^{y^2} \mathrm{erf}c(y) \tag{9}$$

with $y = -\exp(-\mathrm{i}\pi/4) q t^{1/2}$ and $q = k_r$ or $-k_r^*$. Using the previous expression for the argument y one can easily verify that $A(t)$, given by equation (8), fulfils time-reversal invariance, namely,

$$A(t) = A^*(-t). \tag{10}$$

The properties and evaluation of the M-functions follow by noting that $M(y) = w(\mathrm{i}y)/2$. In general $w(z) = \exp(-z^2)\ \mathrm{erfc}(-\mathrm{i}z)$ [10]. In particular $M(0) = 1/2$. The M-function satisfies the symmetry relation $M(y) = \mathrm{e}^{y^2} - M(-y)$, provided its argument fulfils $\pi/2 < \arg(y) < 3\pi/2$ [10]. Hence,

$$M(k_r, t) = \mathrm{e}^{-\mathrm{i}k_r^2 t} - M(-k_r, t). \tag{11}$$

Using the above expression in equation (8) gives

$$A(t) = D_r \mathrm{e}^{-\mathrm{i}k_r^2 t} - [D_r M(-k_r, t) - D_r^* M(-k_\mathrm{r}^*, t)]. \tag{12}$$

The above expression exhibits explicitly the exponential and non-exponential contributions to decay of the survival amplitude. One sees that purely exponential decay is obtained by taking $D_r = 1$ and by neglecting the term within the square brackets on the right-hand side of equation (12). Clearly this last term is the one responsible for the non-exponential contributions to decay at both short and long times.

The coefficient D_r appearing in equations (7), (8) and (12) may be obtained from two general conditions. The first condition follows from equation (6) by noting that $A(k = 0) = 0$. This holds provided the propagator either vanishes or goes to a constant at $k = 0$. Otherwise the propagator would diverge to infinity, a physically unsatisfactory and pathological behaviour. Hence from equation (7) we obtain

$$\frac{D_r}{k_r} - \frac{D_r^*}{k_r^*} = 0. \tag{13}$$

[3] In previous work, García-Calderón [7] obtained equation (8) from a general expression of the survival amplitude, for finite-range interactions, that involves an infinite expansion in terms of resonant states and S-matrix complex poles, under the assumption that the initial state has a vanishing overlap with all resonant states except one. Equation (8) has also been obtained in the framework of a schematic theory of nuclear reactions by Moshinsky [11] and in a model for separable interactions in momentum space by Muga *et al* [12]. However, neither of the last two works analyses the single-level resonance decay formula in the full time interval.

The second condition follows from the initial condition $A(t = 0) = 1$, and implies from equations (8) or (12) that

$$\frac{1}{2}(D_r + D_r^*) = 1. \tag{14}$$

It then follows immediately from equations (13) and (14) that D_r is given by

$$D_r = 1 - \mathrm{i}\frac{\beta_r}{\alpha_r} \tag{15}$$

that depends only on the real and imaginary values of the complex pole $k_r = \alpha_r - \mathrm{i}\beta_r$.

2.1. Short times

The short time behaviour of the single resonance survival amplitude may be readily obtained by expanding the M-functions in equation (7) according to the formula [10], $M(q,t) = \sum_s^\infty (-y)^s/[2\Gamma(s/2+1)]$ with y as defined above. It then follows that the leading term in t reads [7]

$$A(t) \approx 1 - \left(\frac{\mathrm{i}}{\pi}\right)^{1/2} \frac{\Gamma_r}{\alpha_r} t^{1/2} \tag{16}$$

where $E_r = \epsilon_r - \mathrm{i}\Gamma_r/2 = k_r^2$. Hence the corresponding survival probability is given by

$$S(t) \approx 1 - \left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma_r}{\alpha_r} t^{1/2}. \tag{17}$$

The above $t^{1/2}$ behaviour at short times contrasts with the short time t^2-dependence of the survival probability commonly considered. This has been examined and clarified by Muga *et al* [12]. The essential point is that a t^2-dependence requires that the first and second energy moments in the expansion of $\exp(-\mathrm{i}Ht)$, with H the complete Hamiltonian, exist. On the other hand, if the mean energy of the initial state is infinite, a $t^{1/2}$ dependence is feasible. The physical realizability of states with infinite first or second moments is a controversial subject [13]. In our approach the initial state is not specified. However we can convince ourselves that the first and second moments are infinite by noting that $A(t)$ may be written as [12]

$$A(t) = \langle\psi|\exp(-\mathrm{i}Ht)|\psi\rangle = 1 - \frac{\mathrm{i}t}{\hbar}\langle\psi|H|\psi\rangle - \frac{t^2}{2\hbar^2}\langle\psi|H^2|\psi\rangle + \cdots. \tag{18}$$

Using the above equation one may write the first and second moments, $\langle\psi|H|\psi\rangle$ and $\langle\psi|H^2|\psi\rangle$, in terms of the time derivatives of the survival amplitude $A(t)$ as

$$\langle\psi|H|\psi\rangle = -\frac{\hbar}{\mathrm{i}}\left[\frac{\mathrm{d}A\,(t)}{\mathrm{d}t}\right]_{t=0} \tag{19}$$

and

$$\langle\psi|H^2|\psi\rangle = -\hbar^2\left[\frac{\mathrm{d}^2A(t)}{\mathrm{d}t^2}\right]_{t=0}. \tag{20}$$

From equation (16) it follows that at short times $\mathrm{d}A/\mathrm{d}t \sim t^{-1/2}$ and $\mathrm{d}^2A/\mathrm{d}t^2 \sim t^{-3/2}$. Hence using these expressions, as appropriate, on the right-hand side of equations (19) and (20) yields an infinite value for both the first and second moments.

2.2. Long times

To analyse the long time behaviour of the single-resonance survival amplitude it is more appropriate to consider equation (12) since it explicitly exhibits the exponential decay contribution. At very long times, once the exponentially decaying contribution becomes negligible, the behaviour of the survival amplitude follows from the asymptotic expansion of the M functions [10], $M(q,t) \sim 1/(qt^{1/2}) + 1/(q^3t^{3/2}) + \cdots$ ($q = k_r$ or $-k_r^*$). It turns out that the coefficient of the leading term in inverse powers of time for $A(t)$, that goes as $t^{-1/2}$, is precisely that given by equation (13) and hence vanishes exactly. Consequently,

$$A(t) \approx \mathrm{e}^{-\mathrm{i}\epsilon_r t}\mathrm{e}^{-\Gamma_r t/2} + \frac{\mathrm{i}}{4(\pi\mathrm{i})^{1/2}}\,\frac{\Gamma_r}{\alpha_r(\epsilon_r^2+\Gamma_r^2)}\frac{1}{t^{3/2}}. \tag{21}$$

If $\epsilon_r \gg \Gamma_r$ one may neglect Γ^2 in the denominator of equation (21) to obtain the expression for the asymptotic long time non-exponential contribution to decay derived by Goldberger and Watson [14]. One may write the time in lifetime units, $\tau = \Gamma_r t$, to obtain the survival amplitude in terms of the parameter $R = \epsilon_r/\Gamma_r$, namely,

$$A(t) \approx \mathrm{e}^{-\mathrm{i}(R-\mathrm{i}/2)\tau} + F(R)\frac{1}{\tau^{3/2}} \tag{22}$$

with $F(R) = \{2(\mathrm{i}\pi)^{1/2}[2R + (4R^2+1)^{1/2}]^{1/2}(R^2+1)\}^{-1}$. For very large values of R we have $F(R) \approx [4(\mathrm{i}\pi)^{1/2}R^{5/2}]^{-1}$ and hence the survival probability $S(\tau)$ becomes the simple expression

$$S(\tau) \approx \mathrm{e}^{-\tau} + \left(\frac{1}{16\pi R^5}\right)\tau^{-3}. \tag{23}$$

It might be of interest to mention that the long time behaviour of $A(t)$ as $t^{-3/2}$ in fact follows from a general argument that involves the steepest-descent method to asymptotically evaluate equation (5), that holds even if the explicit form of $A(k)$ is not known [7, 8]. This involves the deformation of the contour c in the above equation to a line 45° off the real k-axis and the Taylor expansion of $A(k)$ around the saddle point $k = 0$ of the exponential term in the integrand of equation (5). Hence, around $k = 0$, $A(k) \approx A(0) + kA'(0) + k^2A''(0) + \cdots$, where $A'(0)$ and $A''(0)$ denote, respectively, the first and second derivatives with respect to k evaluated at $k = 0$. The first and second terms in the above Taylor expansion yield a vanishing contribution to the asymptotic value of $A(t)$ because $A(0) = 0$ and $kA'(0)$ makes the integrand an odd function of k. It is then straightforward to see that the term $k^2A''(0)$ yields the $t^{-3/2}$ asymptotic behaviour of $A(t)$. Using equation (7) to evaluate $A''(0)$ reproduces the corresponding coefficient in equation (21).

We end this section by emphasizing the relevance of the pole $-k_r^*$ seated on the third quadrant of the complex k-plane. It plays a crucial role in fulfilling time-reversal invariance and in the derivation of the short- and long time expressions for the survival probability. We also emphasize that some relevant previous results to long time non-exponential contributions to decay were derived in the limit $R \gg 1$, whereas our approach holds for any value of R in the full time interval.

3. Analysis of the survival probability

The expression for $A(t)$ given by equation (12) depends only on the resonance parameters α_r and β_r. As mentioned above if we work with lifetime units ($\tau = \Gamma_r t$) the relevant input is the ratio of the resonance energy ϵ_r to the resonance width Γ_r, namely, $R = \epsilon_r/\Gamma_r$.

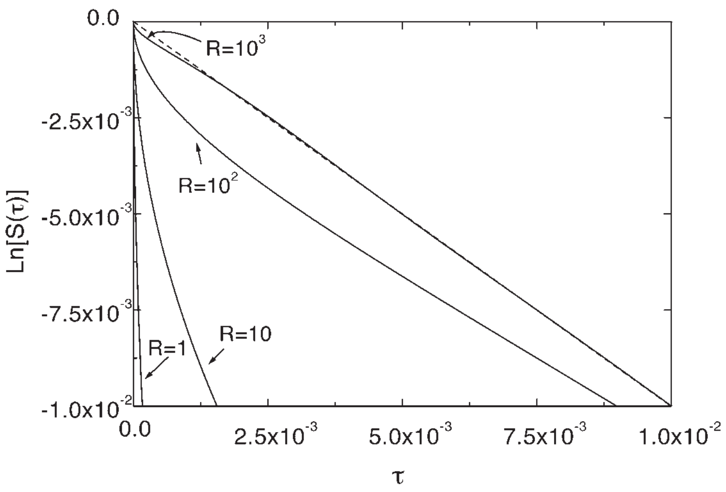

Figure 1. Plot of $\ln S(\tau)$ against τ, with $S(\tau)$ the survival probability (calculated using equation (12)) and τ the number of lifetimes, in the short time regime for several values of $R = \epsilon_r/\Gamma_r$. The dashed line that represents purely exponential decay is to guide the eye (see text).

In order to exhibit the very short time behaviour of the survival probability $S(\tau) = |A(\tau)|^2$, in figure 1 we plot $\ln S(\tau)$ against τ for different values of R. At early times all curves behave according to equation (17). The straight line corresponds to purely exponential decay and is to guide the eye. One may estimate the timescale when the short time behaviour of the survival probability for a single resonance becomes comparable to purely exponential decay, given by $S_{\rm ed} = |D_r|^2 \exp(-\Gamma_r t)$. Using equation (17) this of course occurs when $1 - (2/\pi)^{1/2}(\Gamma_r/\alpha_r)t^{1/2} \approx |D_r|^2 \exp(-\Gamma_r t)$. Provided $\alpha_r > \beta_r$, one may write the resulting timescale in terms of the parameters τ and R (with $R \geqslant 1$) as the simple expression

$$\tau_{\rm S} \approx \left(\frac{4}{\pi}\right) \frac{1}{R^2 + R + 1/4}. \qquad (24)$$

In order to exemplify the long time behaviour of the survival probability, figure 2 exhibits a plot of $\ln S(\tau)$ against τ. One sees that all curves have a similar structure. The values next to each curve represent the corresponding values of R. One may distinguish three regions: first, there is an interval along which the decay is purely exponential (the short-time region is so small that it cannot be perceived), then an oscillatory region appears. This region represents a transient regime where the transition from exponential to non-exponential decay occurs. Note that it may involve several lifetimes. Thereafter, in the third region, the inverse power law τ^{-3} dominates the decay. Note also that as R increases, the deviation from exponential decay to non-exponential decay occurs at a larger number of lifetimes. A simple expression for the onset from exponential to non-exponential decay may be obtained by noting that the survival amplitude $S(t)$ may be expressed as the sum of three quantities describing, respectively, the above three regions. Equation (12) may be rewritten as $A = A_{\rm ed} + A_{\rm ned}$ with $A_{\rm ed} = D_r \exp(-{\rm i}k_r^2 t)$ and $A_{\rm ned} = [DM(-k_r, t) + D_r^* M(-k_r^*, t)]$. Clearly, the survival probability for the single resonance is given by

$$S(t) = S_{\rm ed}(t) + S_{\rm ned}(t) + S_{\rm int}(t) \qquad (25)$$

where $S_{\rm ed} = |A_{\rm ed}|^2$, $S_{\rm ned} = |A_{\rm ned}|^2$, and $S_{\rm int} = 2{\rm Re}\,[A_{\rm ed}^* A_{\rm ned}]$. The exponential decay occurs when the term $S_{\rm ed}$ is the dominant contribution to S. The non-exponential nature of the decay

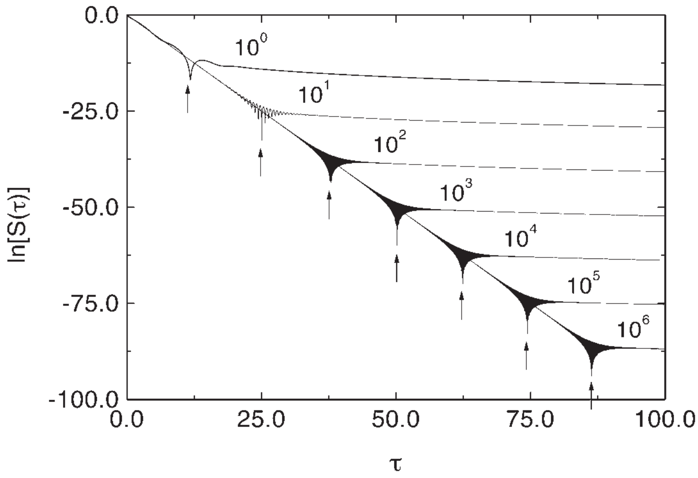

Figure 2. Plot of ln S against τ, with $S(\tau)$ the survival probability (calculated from equation (12)), τ the number of lifetimes, in the long time regime for several values of $R = \epsilon_r/\Gamma_r$. The straight line refers to exponential decay. The arrows indicate the values τ_L where S_{ed} and S_{ned} cross each other (see text).

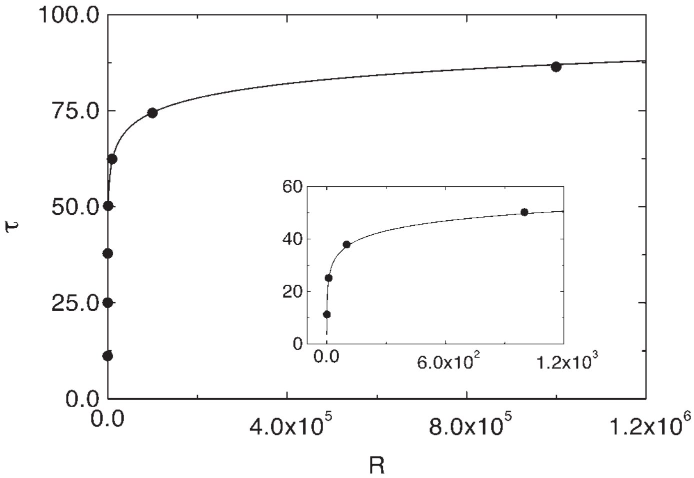

Figure 3. Plot of τ_L, given by equation (26), against $R = \epsilon_r/\Gamma_r$ (continuous curve). The full circles correspond to the values of τ_L indicated by the arrows in figure 2. The inset exhibits a similar calculation for smaller values of R (see text).

arises from the S_{ned} term, which at very long times follows the well known t^{-3} behaviour. The interference contribution S_{int} is an oscillatory function of time. When S_{ed} becomes comparable to S_{ned} the transition from exponential to non-exponential decay occurs as described above. From these considerations we may establish a criterion for the onset of non-exponential decay as the value of the time, τ_L, when S_{ed} and S_{ned} cross each other. For the cases considered in figure 1, the values of τ_L obtained with this criterion are plotted against R in figure 3 (full circles). The distribution of these points suggests a logarithmic R-dependence of the type

$$\tau_L = A\ln(R) + B. \tag{26}$$

The values of the constants $A = 5.41$ and $B = 12.25$ are obtained from a fitting procedure. A plot of τ_L against R using the above formula is shown by the solid curve in figure 3. It can be noted that an excellent description is obtained in the broad interval considered. The inset shows an analogous calculation for smaller values of R. Equation (26) provides a more precise estimate for the long time non-exponential contribution to decay than the rough estimate given by Winter [16], $\tau_L = K \ln(R)$, with K a numerical factor, in the range from two to ten, that depends on the shape of the low-energy end of the spectrum. The problem of analytically finding the explicit dependence of τ_L with respect to R may be a difficult task, except for the asymptotic case, which enables us to use the simple expression given by equation (23). From that one can derive an expression for τ_L,

$$\tau_L - 3\ln(\tau_0) = 5\ln(R) + \ln(16\pi). \tag{27}$$

It is not difficult to convince oneself that for large values of R the two formulae give essentially the same numerical results. For example for $R = 1.0 \times 10^{20}$ equation (26) gives $\tau_L = 261.39$ whereas equation (27) yields $\tau_L = 250.75$. The discrepancy is less than 0.5%.

We end this section by emphasizing that the analysis for the survival probability for the single-level resonance formula provides three relevant timescales: the lifetime t_l, the short timescale τ_S, given by equation (24), and the long timescale τ_L, given by equation (26). It might be of interest to compare our results with the general analysis of non-exponential decay made by Greenland [15]. He establishes a memory timescale that has no counterpart in our approach, since we do not consider any interaction that couples the decaying state to the continuum. In addition to the lifetime timescale he also identifies a long timescale, going as $\ln(R)$, that essentially coincides with ours in the limit $R \gg 1$. However for short times Greenland has $\tau_S^G \sim 1/R$, different from our result given by equation (24). Note however that for $R \sim 1$, $\tau_S \approx \tau_S^G/2$. In general the difference between these timescales originates in the different early-time behaviours of the survival probability: in our case we obtain a $t^{1/2}$-dependence whereas Greenland assumes a t^2-dependence.

4. Comparison and applications

Let us compare the predictions of our approach with a numerical example involving a finite-range potential. The survival amplitude for this problem has been solved exactly and may be expressed as a sum of products of M-functions and expansion coefficients (that consist of the overlap of the arbitrary initial state with the resonant states of the system) [7, 8], namely,

$$A(t) = \sum_{r=1}^{\infty} [D_r M(k_r, t) + D_r^* M(-k_r^*, t)] \tag{28}$$

where the expansion coefficients $D_r = C_r \overline{C}_r$ with $C_r = \int_0^R \psi(r, 0) u_(r)\, \mathrm{d}r$ and $\overline{C}_r = \int_0^R \psi^*(r, 0) u_r(r)\, \mathrm{d}r$, with the set of functions $u_r(r)$, correspond to the resonant states of the system [7, 8]. These coefficients have some useful properties; in particular they fulfil the sum rules:

$$\sum_r^{\infty} \left(\frac{D_r}{k_r} - \frac{D_r^*}{k_r^*} \right) = 0 \tag{29}$$

and

$$\tfrac{1}{2} \sum_r^{\infty} (D_r + D_r^*) = 1. \tag{30}$$

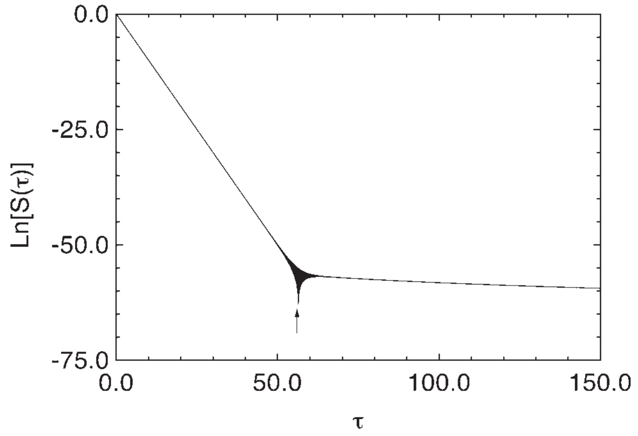

Figure 4. Plot of $\ln S(\tau)$ against τ, with $S(\tau)$ the survival probability and τ the number of lifetimes, for the value $R = 3257.33$ that corresponds to the first resonance pole in the delta-shell potential. The value $\tau = 56$ obtained using equation (26) is indicated by the arrow (see the text).

As mentioned previously equations (28)–(30) become equations (8), (13) and (14) provided it is assumed that the overlap of the initial state $\psi(r, 0)$ with all states $u_n(r)$ vanishes except for one of them, say the rth [7].

Our example corresponds to the delta-shell potential $V(r) = \lambda\delta(r - a)$, considered in detail in [7, 8]. We refer the reader to the calculation for the time evolution of the survival probability displayed in figure 1 of [8]. It is worth emphasizing that the initial state there, $\psi(r, 0) = 2^{1/2}\sin(\pi/R)r$, is close to the resonant pole with the longest lifetime, namely $k_1 = 3.126 - \mathrm{i}0.000\,24$. The parameters of the potential are $\lambda = 200$ and $a = 1$ (the units are $\hbar = 2m = 1$). The above value of k_1, that leads to $R = 3257.33$, can be put into our expression for the single-level survival amplitude given by equations (8) or (12) to allow us to calculate the survival probability as a function of the number of lifetimes τ. This is displayed in figure 4. The deviation for non-exponential decay occurs for $\tau = 56$, indistinguishable from the result for the full calculation involving many resonance terms displayed in figure 1 of [8]. Note that the one-term approximation to equation (28), in view of equations (29) and (30), does not lead to the $t^{-3/2}$ behaviour for the survival amplitude, even if Re D_1 is close to unity.

Let us now apply the single-level formula to some real systems. As mentioned previously so far the only experimental evidence for the deviation of the exponential decay law has been reported in the short time regime [6] for a system consisting of ultra-cold sodium atoms trapped in an accelerating optical potential created by two counter-propagating lasers. The resulting potential is analogous to that for electrons moving in a periodic lattice with a dc electric field and hence it exhibits a band structure. It follows from the above considerations that the decay process cannot be ascribed to a single resonance. Indeed, the decay process has been visualized as Landau–Zener tunnelling between Bloch bands, providing a reasonable agreement with experiment [17]. Another experiment in the short time regime is the search for nuclear radioactive decay made by Norman *et al*. They studied the β decay of ^{60}Co to search for deviations of the exponential decay law up to times as small as $10^{-4}\,t_{1/2}$ with negative results. In this case the resonance parameters are $\epsilon_r = 0.3193$ MeV and $\Gamma_r = 2.744 \times 10^{-30}$ MeV, so $R = 1.16 \times 10^{29}$. Equation (24) predicts non-exponential decay at $\tau_S \sim 10^{-58}$ of a lifetime. This is an extremely small quantity, much smaller than the tested fraction of a lifetime. A similar result is obtained for an improved test using ^{40}K [5] where the validity of the exponential decay law was extended up to $10^{-10}\,t_{1/2}$. Here $R = 1.38 \times 10^{38}$ and hence in this case $\tau_S \sim 10^{-76}$ of a lifetime, much smaller than the previous example.

Regarding the search on non-exponential contributions to decay at long times we also refer to the work by Norman *et al* [4]. These authors studied the decay law of ^{56}Mn in the interval $0.3 < \tau < 45$ and detected no deviation from the exponential decay law. For the radioactive decay of ^{56}Mn, with $\epsilon_r = 2.81$ MeV and $\Gamma_r = 7.09 \times 10^{-26}$ MeV, associated with the lifetime $2.576\,\mathrm{hr}/\ln 2$, gives $R = 3.96 \times 10^{25}$ and hence, using equation (26), the number of lifetimes where the deviation from the exponential decay law would occur is around $\tau_L = 331$. This is larger than the estimate of $\tau_L \approx 200$ given by Winter [16], though it lies within the limits given by his formula, as discussed in the previous section. Certainly our prediction is far from the time interval considered in the experiment by Norman and co-workers. The results of figure 2 suggest that values of R of the order of unity may be more adequate to observe non-exponential decay. An appealing example is the decay of the first unbound state of ^{5}He [18]. There $R \sim 1$ and from equation (26) one would expect the onset of non-exponential decay to occur for $\tau_L \sim 12$. However the lifetime of the state is extremely short, $t_l \approx 10^{-22}$ s, and hence a measurement of non-exponential decay would require us to measure times of the order of 10^{-21} s, that are beyond present-day technologies. This example suggests that in addition to $R \sim 1$ it is also required that the lifetime t_l of the system be a measurable timescale.

5. Concluding remarks

We have considered an exact single-level resonance formula for the survival amplitude (equation (12)) and consequently of the survival probability, to describe the transition from exponential to non-exponential decay for both the short time and the long time regimes, given respectively by the timescales τ_S (equation (24)) and τ_L (equation (26)). The only inputs required in these expressions are the ratio $R = \epsilon_r/\Gamma_r$ and the number or fraction of lifetimes τ involved. The single-level resonance formula could be useful for determining appropriate resonance parameters in the search for evidence of non-exponential decay in quantum systems. Cases where $R \sim 1$ seem to be of particular interest. There $\tau_S \sim 1/2$ and $\tau_L \sim 12$. In order to detect the non-exponential contribution, and in addition to have initially a sufficient number of decaying systems, the lifetime of the system must be measurable with present-day technologies. It has been recognized that Nature does not seem to favour the occurrence of the correct combination of circumstances to exhibit the non-exponential contributions to decay. This suggests resorting to artificial quantum structures, where one may design and control the relevant parameters of the system, to seek verification of this old prediction of quantum mechanics.

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